

Static Fracture Analysis of Composites

4.1 INTRODUCTION

4.2 ANISOTROPIC ELASTICITY

4.2.1 Elasticity solution

The general form of an anisotropic stress–strain relationship can be defined in terms of the 4th order material modulus and compliance tensors, $\mathbf{D}$ and $\mathbf{C}$, respectively,

$$\boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon} \quad (4.1)$$

$$\boldsymbol{\varepsilon} = \mathbf{C} : \boldsymbol{\sigma} \quad (4.2)$$

or in the component forms of

$$\sigma_{ij} = D_{ijkl} \varepsilon_{kl} \quad i, j = 1, 2, 3 \quad (4.3)$$

$$\varepsilon_{ij} = C_{ijkl} \sigma_{kl} \quad i, j = 1, 2, 3 \quad (4.4)$$

Equations (4.3) and (4.4) can also be written in a more appropriate component form,

$$\sigma_i = d_{ij} \varepsilon_j \quad i, j = 1, 2, 3, \dots, 6 \quad (4.5)$$

$$\varepsilon_i = c_{ij} \sigma_j \quad i, j = 1, 2, 3, \dots, 6 \quad (4.6)$$

For an orthotropic material which has three mutually orthogonal planes of elastic symmetry, Equations (4.5) and (4.6) are reduced to:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & 0 & 0 & 0 \\ d_{12} & d_{22} & d_{23} & 0 & 0 & 0 \\ d_{13} & d_{23} & d_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \quad (4.7)$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \quad (4.8)$$

where

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} \quad (4.9)$$

Equations (4.7) and (4.8) are valid for arbitrarily selected coordinate systems. Further simplification is obtained for the principal directions of orthotropy. The components d_{ij} of Eq. (4.7) can then be defined as (Bower, 2012),

$$\begin{bmatrix} E_1(1-\nu_{23}\nu_{32})\Delta & E_1(\nu_{12}+\nu_{31}\nu_{23})\Delta & E_1(\nu_{31}+\nu_{21}\nu_{31})\Delta & 0 & 0 & 0 \\ E_1(\nu_{12}+\nu_{31}\nu_{23})\Delta & E_2(1-\nu_{13}\nu_{31})\Delta & E_2(\nu_{32}+\nu_{12}\nu_{31})\Delta & 0 & 0 & 0 \\ E_1(\nu_{31}+\nu_{21}\nu_{31})\Delta & E_2(\nu_{32}+\nu_{12}\nu_{31})\Delta & E_3(1-\nu_{12}\nu_{21})\Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_{12} \end{bmatrix} \quad (4.10)$$

where

$$\frac{1}{\Delta} = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13} \quad (4.11)$$

and the components c_{ij} of Eq. (4.8) can be written as,

$$\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_{12}} \end{bmatrix} \quad (4.12)$$

with the following additional relations to keep the number of independent constants unchanged:

$$E_1 \nu_{21} = E_2 \nu_{12} \quad (4.13)$$

$$E_2 \nu_{32} = E_3 \nu_{23} \quad (4.14)$$

$$E_3 \nu_{13} = E_1 \nu_{31} \quad (4.15)$$

These components can be further simplified for plane stress and strain states. The components d_{ij} for a generalized plane stress problem is directly computed from (4.10)

$$\begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{22} & 0 \\ 0 & 0 & d_{66} \end{bmatrix} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & \mu_{12} \end{bmatrix} \quad (4.16)$$

and for generalized plane strains, (Garcia-sanchez *et al.*, 2008)

$$\begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{22} & 0 \\ 0 & 0 & d_{66} \end{bmatrix} = \begin{bmatrix} E_1 \Delta (1-\nu_{12}\nu_{21}) & E_1 \Delta (\nu_{21} + \frac{E_2}{E_1} \nu_{13}\nu_{32}) & 0 \\ E_1 \Delta (\nu_{21} + \frac{E_2}{E_1} \nu_{13}\nu_{32}) & E_2 \Delta (1-\nu_{13}\nu_{31}) & 0 \\ 0 & 0 & \mu_{12} \end{bmatrix} \quad (4.17)$$

The components of the compliance matrix c_{ij} for plane stress problems are also directly determined from (4.12). A simple rule for change of parameters can then be used for the plane stress problems (Ghorashi *et al.*, 2011a),

$$c_{ij} = \begin{cases} c_{ij} & i, j = 1, 2, 6 \text{ plane stress} \\ c_{ij} - \frac{c_{i3}c_{j3}}{c_{33}} & i, j = 1, 2, 6 \text{ plane strain} \end{cases} \quad (4.18)$$

4.2.2 Anisotropic stress functions

The Airy stress function is limited to isotropic problems. For an extension to more complex problems, including anisotropic problems, the stress function $\Phi(x, y)$ can be written as

$$\Phi(x, y) = 2\text{Re} [\Phi_1(z_1) + \Phi_2(z_2)] \quad (4.19)$$

where $\Phi_1(z_1)$ and $\Phi_2(z_2)$ are arbitrary functions of $z_1 = x + s_1 y$ and $z_2 = x + s_2 y$, respectively. Combining the definition of stress components from the Airy stress function and satisfying the compatibility equation, the following relation is obtained for anisotropic solids in the absence of body forces:

$$c_{22} \frac{\partial^4 \Phi}{\partial x^4} - 2c_{26} \frac{\partial^4 \Phi}{\partial x^3 \partial y} + (2c_{12} + c_{66}) \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} - 2c_{16} \frac{\partial^4 \Phi}{\partial x \partial y^3} + c_{22} \frac{\partial^4 \Phi}{\partial y^4} = 0 \quad (4.20)$$

where coordinates (x, y) coincides with material axes (1,2) or (x_1, y_1) . Eq. (4.20) reduces to the following simplified equation for isotropic problems,

$$\frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = \nabla^2 (\nabla^2 \Phi) = 0 \quad (4.21)$$

The characteristic equation of the homogenous partial differential equation (4.20) is

$$c_{11}s^4 - 2c_{16}s^3 + (2c_{12} + c_{66})s^2 - 2c_{26}s + c_{22} = 0 \quad (4.22)$$

Lekhnitski (1968) discussed the availability and conditions for the roots of Eq. (4.22). Here, only the two isotropic and orthotropic cases are considered. For an isotropic case, the roots are $s_1 = s_2 = i$ and $\bar{s}_1 = \bar{s}_2 = -i$, whereas for an orthotropic material with axes of orthotropy (1,2) coinciding Cartesian (x, y) axes, $c_{16} = c_{26} = 0$, the characteristic Eq. (4.22) is reduced to:

$$c_{11}s^4 + (2c_{12} + c_{66})s^2 + c_{22} = 0 \quad (4.23)$$

with pure imaginary roots s_1 and s_2 (Sadd, 2005). Finally, the stress components are defined from the second derivatives of the complex stress function Φ_i^* :

$$\sigma_x = 2\text{Re} [s_1^2 \Phi_1''(z_1) + s_2^2 \Phi_2''(z_2)] \quad (4.24)$$

$$\sigma_y = 2 \operatorname{Re} [\Phi_1''(z_1) + \Phi_2''(z_2)] \quad (4.25)$$

$$\sigma_{xy} = -2 \operatorname{Re} [s_1 \Phi_1''(z_1) + s_2 \Phi_2''(z_2)] \quad (4.26)$$

and the displacements are obtained from the first derivatives of the complex stress function, Φ_i' :

$$u_x = 2 \operatorname{Re} [p_1 \Phi_1'(z_1) + p_2 \Phi_2'(z_2)] \quad (4.27)$$

$$u_y = 2 \operatorname{Re} [q_1 \Phi_1'(z_1) + q_2 \Phi_2'(z_2)] \quad (4.28)$$

where

$$p_i = c_{11}s_i^2 + c_{12} - c_{16}s_i \quad i = 1, 2 \quad (4.29)$$

$$q_i = c_{12}s_i + \frac{c_{22}}{s_i} - c_{26} \quad i = 1, 2 \quad (4.30)$$

4.3 ANALYTICAL SOLUTIONS FOR NEAR CRACK TIP

Assume an anisotropic body containing a crack is subjected to arbitrary forces with general displacement and traction boundary conditions. Global Cartesian coordinates are (X, Y) and local Cartesian coordinate (x, y) and local polar coordinate (r, θ) are defined on the crack tip as illustrated in Fig. 4.1.

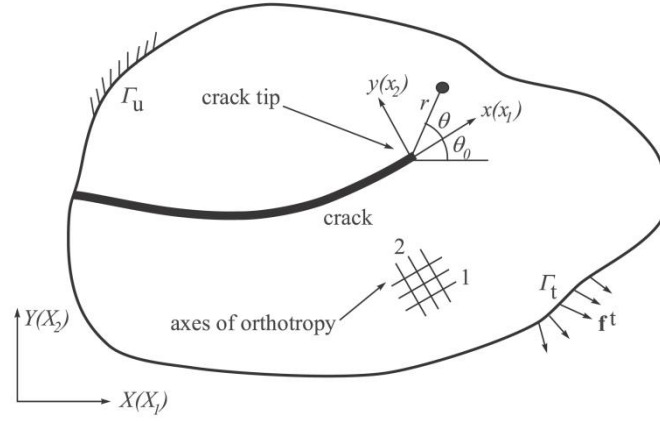


Figure 4.1 Local coordinates at both crack tips.

4.3.1 The general solution

The stress-strain equations for an orthotropic medium, with axes of elastic symmetry $(1,2) = (x_1, x_2)$ coincident with the Cartesian coordinates axes $(x_1, x_2) = (x, y)$, can be defined in terms of the material modulus coefficients d_{ij} :

$$\sigma_x = d_{11} \frac{\partial u_x}{\partial x} + d_{12} \frac{\partial u_y}{\partial y} \quad (4.31)$$

$$\sigma_y = d_{12} \frac{\partial u_x}{\partial x} + d_{22} \frac{\partial u_y}{\partial y} \quad (4.32)$$

$$\sigma_{xy} = d_{66} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \quad (4.33)$$

The set of equations for an inplane elastostatic problem can be expressed as:

$$d_{11} \frac{\partial^2 u_x}{\partial x^2} + d_{66} \frac{\partial^2 u_x}{\partial y^2} + (d_{12} + d_{66}) \frac{\partial^2 u_y}{\partial x \partial y} = 0 \quad (4.34)$$

$$d_{66} \frac{\partial^2 u_y}{\partial x^2} + d_{22} \frac{\partial^2 u_y}{\partial y^2} + (d_{12} + d_{66}) \frac{\partial^2 u_x}{\partial x \partial y} = 0 \quad (4.35)$$

Now recalling the characteristic equation (4.22) of the governing fourth-order partial differential equation for the general case of Figure 4.1, the roots of Eq. (4.22) are always complex or purely imaginary ($s_k = s_{kx} + is_{ky}$, $k=1,2$) and occur in conjugate pairs as s_1 , $\bar{s}_1$ and s_2 , $\bar{s}_2$ (Lekhnitskii, 1968).

Sih *et al.* (1965) derived the two-dimensional displacement and stress fields in the vicinity of the crack tip in an infinite anisotropic plate by means of the stress function (4.19), analytical functions and complex variables, $z_k = x + s_k y$, $k=1,2$. The displacement components for pure mode I are defined as,

$$u_1^I = K_I \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{1}{s_1 - s_2} \left[s_1 p_2 (\cos \theta + s_2 \sin \theta)^{\frac{1}{2}} - s_2 p_1 (\cos \theta + s_1 \sin \theta)^{\frac{1}{2}} \right] \right\} \quad (4.36)$$

$$u_2^I = K_I \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{1}{s_1 - s_2} \left[s_1 q_2 (\cos \theta + s_2 \sin \theta)^{\frac{1}{2}} - s_2 q_1 (\cos \theta + s_1 \sin \theta)^{\frac{1}{2}} \right] \right\} \quad (4.37)$$

and the stress components are

$$\sigma_{11}^I = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{s_1 s_2}{s_1 - s_2} \left(\frac{s_2}{(\cos \theta + s_2 \sin \theta)^{\frac{1}{2}}} - \frac{s_1}{(\cos \theta + s_1 \sin \theta)^{\frac{1}{2}}} \right) \right] \quad (4.38)$$

$$\sigma_{22}^I = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left(\frac{s_1}{(\cos \theta + s_2 \sin \theta)^{\frac{1}{2}}} - \frac{s_2}{(\cos \theta + s_1 \sin \theta)^{\frac{1}{2}}} \right) \right] \quad (4.39)$$

$$\sigma_{12}^I = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{s_1 s_2}{s_1 - s_2} \left(\frac{-1}{(\cos \theta + s_2 \sin \theta)^{\frac{1}{2}}} + \frac{1}{(\cos \theta + s_1 \sin \theta)^{\frac{1}{2}}} \right) \right] \quad (4.40)$$

and in the same way, for pure mode II (skew-symmetric loading), the displacement and stress fields are defined as:

$$u_1^{II} = K_{II} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{1}{s_1 - s_2} \left[p_2 (\cos \theta + s_2 \sin \theta)^{\frac{1}{2}} - p_1 (\cos \theta + s_1 \sin \theta)^{\frac{1}{2}} \right] \right\} \quad (4.41)$$

$$u_2^{II} = K_{II} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{1}{s_1 - s_2} \left[q_2 (\cos \theta + s_2 \sin \theta)^{\frac{1}{2}} - q_1 (\cos \theta + s_1 \sin \theta)^{\frac{1}{2}} \right] \right\} \quad (4.42)$$

and the stress components are

$$\sigma_{11}^{\text{II}} = \frac{K_{\text{II}}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left(\frac{s_2^2}{(\cos \theta + s_2 \sin \theta)^{1/2}} - \frac{s_1^2}{(\cos \theta + s_1 \sin \theta)^{1/2}} \right) \right] \quad (4.43)$$

$$\sigma_{22}^{\text{II}} = \frac{K_{\text{II}}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left(\frac{1}{(\cos \theta + s_2 \sin \theta)^{1/2}} - \frac{1}{(\cos \theta + s_1 \sin \theta)^{1/2}} \right) \right] \quad (4.44)$$

$$\sigma_{12}^{\text{II}} = \frac{K_{\text{II}}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left(\frac{-s_2}{(\cos \theta + s_2 \sin \theta)^{1/2}} + \frac{s_1}{(\cos \theta + s_1 \sin \theta)^{1/2}} \right) \right] \quad (4.45)$$

where K_{I} and K_{II} are the stress intensity factors for modes I and II, respectively, and p_k and q_k are defined in Eqs. (4.29) and (4.30).

The original assumption of coincident material and crack tip local axes in deriving the analytical solutions can be generalized to non-coincident axes for developing the basis functions for orthotropic crack tip enrichment function. The crack tip enrichment functions in polar coordinate system are then obtained from Eqs. (4.36), (4.37), (4.41) and (4.42), as discussed in Section 4.5.3 (Asadpoure and Mohammadi, 2007).

4.3.2 Special solutions for different types of composites

Two types of composites can be distinguished from the roots of the characteristic equation (4.22). Defining

$$a_1 = \frac{(\alpha_1 + \alpha_2 - 4\beta_1\beta_2)}{2} \quad (4.46)$$

$$a_2 = \alpha_1\alpha_2 \quad (4.47)$$

with

$$\alpha_1 = \frac{d_{66}}{d_{11}} \quad \alpha_2 = \frac{d_{22}}{d_{66}} \quad \beta_1 = \frac{d_{12} + d_{66}}{2d_{11}} \quad \beta_2 = \frac{d_{12} + d_{66}}{2d_{66}} \quad (4.48)$$

Then, two different types of orthotropic materials can be defined based on the existence of the real part of the solution: Type I ($a_1 > \sqrt{a_2}, a_2 > 0$) where the characteristic equation has only imaginary roots, and Type II ($|a_1| < \sqrt{a_2}, a_2 > 0$) with complex roots for (4.22), as discussed by Piva and Viola (1988) and Gentilini *et al.* (2004), among the others.

The first type was solved by Carloni and Nobile (2002), Carloni *et al.* (2003), Nobile and Carloni (2005),

$$\begin{aligned}
 u_1 = & \frac{2m_2}{d_{66}(e_1 - e_2)} \sqrt{2lr} \left\{ \sigma_{02} \left[\frac{e_2 \sqrt{g_2(\theta)}}{m_4(m_1 - e_2^2)} \cos \frac{\theta_2}{2} - \frac{e_1 \sqrt{g_1(\theta)}}{m_3(m_1 - e_1^2)} \cos \frac{\theta_1}{2} \right] \right. \\
 & \left. + e_1 e_2 \sigma_{03} \left[\frac{\sqrt{g_2(\theta)}}{m_4(m_1 - e_2^2)} \sin \frac{\theta_2}{2} - \frac{\sqrt{g_1(\theta)}}{m_3(m_1 - e_1^2)} \sin \frac{\theta_1}{2} \right] \right\} \\
 & - \frac{2m_2 e_1 e_2 (\sigma_{02} - e_1 e_2 \sigma_{01})}{d_{66} m_3 m_4 (m_1 - e_1^2)(m_1 - e_2^2)} (l + r \cos \theta) - \frac{m_2 \sigma_{03} (e_1 + e_2)^2}{d_{66} m_3 m_4 (m_1 - e_1^2)(m_1 - e_2^2)} r \sin \theta
 \end{aligned} \quad (4.49)$$

$$\begin{aligned}
 u_2 = & \frac{1}{d_{66}(e_1 - e_2)} \frac{\sqrt{2lr}}{m_3 m_4} \left\{ \sigma_{02} \left[m_3 \sqrt{g_2(\theta)} \sin \frac{\theta_2}{2} - m_4 \sqrt{g_1(\theta)} \sin \frac{\theta_1}{2} \right] \right. \\
 & \left. + \sigma_{03} \left[m_4 e_2 \sqrt{g_1(\theta)} \cos \frac{\theta_1}{2} - m_3 e_1 \sqrt{g_2(\theta)} \cos \frac{\theta_2}{2} \right] \right\} \\
 & + \frac{\sigma_{03} (e_1 + e_2) (m_3 - m_4)}{2 d_{66} m_3 m_4 (e_1 - e_2)} (l + r \cos \theta) \\
 & + \frac{(\sigma_{02} - e_1 e_2 \sigma_{01})}{d_{66} (e_1^2 - e_2^2)} \left(\frac{e_2}{m_3 e_1} - \frac{e_1}{m_4 e_2} \right) \frac{m_2 \sigma_{03} (e_1 + e_2)^2}{d_{66} m_3 m_4 (m_1 - e_1^2)(m_1 - e_2^2)} r \sin \theta
 \end{aligned} \quad (4.50)$$

where m_1, m_2, m_3 and m_4 are coefficients related to material properties (see Carloni *et al.*, 2003 and Mohammadi, 2008), and e_1 and e_2 are defined as (Carloni *et al.*, 2003),

$$e_1 = \left\{ \frac{1}{2} \left[\frac{d_{66}}{d_{11}} + \frac{d_{22}}{d_{66}} - \frac{(d_{12} + d_{66})^2}{d_{11} d_{66}} \right] - \left(\frac{1}{4} \left[\frac{d_{66}}{d_{11}} + \frac{d_{22}}{d_{66}} - \frac{(d_{12} + d_{66})^2}{d_{11} d_{66}} \right]^2 - \frac{d_{22}}{d_{11}} \right)^{1/2} \right\}^{1/2} \quad (4.51)$$

$$e_2 = \left\{ \frac{1}{2} \left[\frac{d_{66}}{d_{11}} + \frac{d_{22}}{d_{66}} - \frac{(d_{12} + d_{66})^2}{d_{11} d_{66}} \right] + \left(\frac{1}{4} \left[\frac{d_{66}}{d_{11}} + \frac{d_{22}}{d_{66}} - \frac{(d_{12} + d_{66})^2}{d_{11} d_{66}} \right]^2 - \frac{d_{22}}{d_{11}} \right)^{1/2} \right\}^{1/2} \quad (4.52)$$

Equations (4.51) and (4.52) were used by Asadpoure (2006) and Asadpoure *et al.* (2007) to derive XFEM crack tip enrichment functions (See Section 4.5.3).

The same problem with the condition of $|a_i| < \sqrt{a_2}$ (type II) was considered by Viola *et al.* (1989). Accordingly, Asadpoure *et al.* (2006) proposed the following enrichment functions for this class of orthotropic composites:

$$\begin{aligned}
u_1 = & -2\beta_1 t_5 + \frac{\beta \sigma_0}{d_{66} k_7} \left\{ t_1 \left[2(a + r \cos \theta) - \sqrt{2ar} \left(\sqrt{g_1(\theta)} \cos \frac{\theta_1}{2} + \sqrt{g_2(\theta)} \cos \frac{\theta_2}{2} \right) \right] \right. \\
& \left. - t_2 \sqrt{2ar} \left(\sqrt{g_1(\theta)} \sin \frac{\theta_1}{2} - \sqrt{g_2(\theta)} \sin \frac{\theta_2}{2} \right) \right\} \\
& + \frac{\beta_1 \tau_0}{d_{66} k_7} \left\{ t_3 \left[X_1 - X_2 + \sqrt{2ar} \left(\sqrt{g_2(\theta)} \cos \frac{\theta_2}{2} - \sqrt{g_1(\theta)} \cos \frac{\theta_1}{2} \right) \right] \right. \\
& \left. - t_4 \left[2Y_1 - \sqrt{2ar} \left(\sqrt{g_1(\theta)} \sin \frac{\theta_1}{2} + \sqrt{g_2(\theta)} \sin \frac{\theta_2}{2} \right) \right] \right\}
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
u_2 = & -h_5 + \frac{\sigma_0}{2d_{66} k_7} \left\{ h_1 \left[(X_1 - X_2) + \sqrt{2ar} \left(\sqrt{g_1(\theta)} \cos \frac{\theta_1}{2} - \sqrt{2ar} \cos \frac{\theta_2}{2} \right) \right] \right. \\
& \left. + h_2 \left[2Y_1 - \sqrt{2ar} \left(\sqrt{g_1(\theta)} \sin \frac{\theta_1}{2} + \sqrt{g_2(\theta)} \sin \frac{\theta_2}{2} \right) \right] \right\} \\
& + \frac{\tau_0}{2d_{66} k_7} \left\{ h_3 \left[2(a + r \cos \theta) - \sqrt{2ar} \left(\sqrt{g_1(\theta)} \cos \frac{\theta_1}{2} + \sqrt{g_2(\theta)} \cos \frac{\theta_2}{2} \right) \right] \right. \\
& \left. + h_4 \sqrt{2ar} \left(\sqrt{g_1(\theta)} \sin \frac{\theta_1}{2} + \sqrt{g_2(\theta)} \sin \frac{\theta_2}{2} \right) \right\}
\end{aligned} \tag{4.54}$$

where

$$\gamma_1 = \left[\frac{1}{2} + \left(\sqrt{a_2} + a_1 \right) \right]^{1/2} \tag{4.55}$$

$$\gamma_2 = \left[\frac{1}{2} + \left(\sqrt{a_2} - a_1 \right) \right]^{1/2} \tag{4.56}$$

$$l^2 = \left(\gamma_1^2 + \gamma_2^2 \right)^{-1} \quad j = 1, 2 \tag{4.57}$$

For definition of other parameters see Viola *et al.* (1989) and Asadpoure *et al.* (2006).

4.4 ORTHOTROPIC MIXED MODE FRACTURE

An orthotropic composite material behaviour can be further idealized in order to simplify the process of crack modeling; a fibre composite structure is assumed as a homogeneous orthotropic continuum, where the crack growth takes place in an idealized material with anisotropic constituents. In this approach, the details of local failures of composite, such as broken fibres or cracked matrix, are not considered and an equivalent orthotropic continuum is adopted.

4.4.1 Energy release rate for anisotropic materials

Sih *et al.* (1965) extended Eqs. (2.136)–(2.137) for anisotropic materials:

$$G_I = -\frac{\pi}{2} K_I c_{22} \operatorname{Im} \left[\frac{K_I (s_1 + s_2) + K_{II}}{s_1 s_2} \right] \quad (4.58)$$

$$G_{II} = \frac{\pi}{2} K_{II} c_{11} \operatorname{Im} [K_{II} (s_1 + s_2) + K_I s_1 s_2] \quad (4.59)$$

Eqs. (4.58) and (4.59) can be conveniently simplified for orthotropic problems with the crack on one plane of material symmetry (Sih *et al.*, 1965):

$$G_I = \pi K_I^2 \sqrt{\frac{c_{11} c_{22}}{2}} \left[\sqrt{\frac{c_{22}}{c_{11}}} + \frac{2c_{12} + c_{66}}{2c_{11}} \right]^{\frac{1}{2}} \quad (4.60)$$

$$G_{II} = \pi K_{II}^2 \frac{c_{11}}{\sqrt{2}} \left[\sqrt{\frac{c_{22}}{c_{11}}} + \frac{2c_{12} + c_{66}}{2c_{11}} \right]^{\frac{1}{2}} \quad (4.61)$$

4.4.2 Anisotropic singular elements

The same idea of singular quarter point finite elements can be extended to anisotropic problems. Saouma and Sikiotis (1986) proposed the following procedure for anisotropic materials in the form of:

$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \frac{1}{\Delta'} \mathbf{B} \begin{Bmatrix} 4(u_{xB} - u_{xD}) - (u_{xC} - u_{xE}) \\ 4(u_{yB} - u_{yD}) - (u_{yC} - u_{yE}) \end{Bmatrix} \sqrt{\frac{\pi}{2L}} \quad (4.62)$$

where u_{ia} is the i -th local crack tip displacement component (x, y) of nodes $a = B, C, D, E$ of the singular quarter point element, as shown in Fig. 4.2, and

$$\mathbf{B} = \begin{bmatrix} \operatorname{Re} \left[\frac{i}{s_1 - s_2} (q_2 - q_1) \right] & \operatorname{Re} \left[\frac{-i}{s_1 - s_2} (p_2 - p_1) \right] \\ \operatorname{Re} \left[\frac{-i}{s_1 - s_2} (s_1 q_2 - s_2 q_1) \right] & \operatorname{Re} \left[\frac{i}{s_1 - s_2} (s_1 p_2 - s_2 p_1) \right] \end{bmatrix} \quad (4.63)$$

$$\Delta' = \det \begin{bmatrix} \operatorname{Re} \left[\frac{i}{s_1 - s_2} (s_1 p_2 - s_2 p_1) \right] & \operatorname{Re} \left[\frac{i}{s_1 - s_2} (p_2 - p_1) \right] \\ \operatorname{Re} \left[\frac{i}{s_1 - s_2} (s_1 q_2 - s_2 q_1) \right] & \operatorname{Re} \left[\frac{i}{s_1 - s_2} (q_2 - q_1) \right] \end{bmatrix} \quad (4.64)$$

and L is the side length of the singular element. Further investigation, however, is required to assess the accuracy of such a complex model.

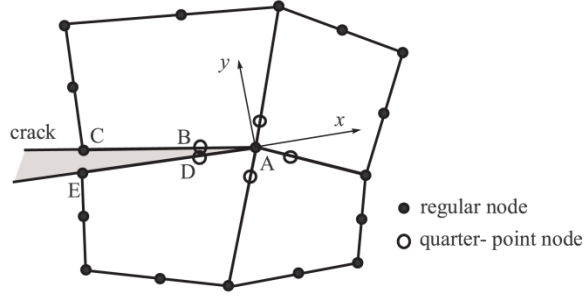


Figure 4.2 Displacement field around the crack tip (Mohammadi, 2008).

4.4.3 SIF calculation by interaction integral

The stress intensity factor (SIF) is one of the important parameters representing fracture properties of a crack tip. Here, the domain integral method, proposed by Kim and Paulino (2002a) is adopted for evaluating the mixed mode stress intensity factors in homogenous orthotropic media.

Fig. 4.3 shows an arbitrary area surrounding a crack tip. The standard path independent J integral for the cracked body is defined as (Rice and Rosengren, 1968):

$$J = \int_{\Gamma} \left(w_s \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) \mathbf{n}_j d\Gamma \quad (4.65)$$

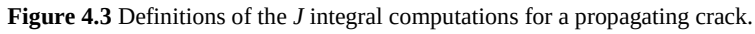
where $\mathbf{n}_j$ is the j^{th} component of the outward unit normal to Γ , δ_{1j} is Kronecker delta, w_s is the strain energy density for linear elastic material, defined in (2.154), and Γ is an arbitrary contour around the crack tip which encloses no other cracks or discontinuities.

Eq. (4.66) is not well suited for the finite element solutions, and an equivalent form of the J integral can be obtained by exploiting the divergence theorem in the form of the domain integral approach (Kim and Paulino, 2003d)

$$J = \int_A \left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - w_s \delta_{1j} \right) \frac{\partial q}{\partial x_j} dA \quad (4.66)$$

where A is an area surrounding the crack tip (the interior region of Γ) and q is a smoothly varying function. Γ is usually assumed as a circular or rectangular area whose centre locates on the crack tip.

As proposed by Dolbow (1999) and Moës *et al.* (1999), a simple function q , varying linearly from $q=1$ at the crack tip to $q=0$ at the exterior boundary Γ , is used in the finite element model (Fig. 4.3). Therefore, the elements away from the boundary can be neglected. It is worth noting that the value within the parentheses in Eq. (4.60) is not necessary to be evaluated in the area that q is constant (and so its gradient vanishes).



Evaluation of (4.65) can be implemented by the Gauss integration rule along the path Γ ,

where W_g^F is the Gauss weighting factor, ng_F is the order of integration and (ξ_g, η_g) are the Gauss points along the contour Γ .

$$q(\mathbf{x}) = \sum N_i(\mathbf{x})q_i \quad (4.68)$$

$$J = \sum_{\substack{\text{elements} \\ \text{in } A}} \left(\sum_{g=1}^{ng_A} \left\{ \left[\left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - w_s \delta_{li} \right) \frac{\partial q}{\partial x_j} \right] \det \mathbf{J} \right\}_g W_g^A \right) \quad (4.69)$$

The domain integral can also be computed using a set of additional integration cells rich in Gauss points. Such a domain can be chosen independent of the finite element mesh and follows the crack tip and its orientations during the crack growth (Gosz *et al.*, 1998 and Gregoire *et al.*, 2007).

The method of interaction integral (Section 2.7.5), based on the definition of an auxiliary state, is used to extract mixed mode stress intensity factors. By combining the actual and auxiliary states for obtaining the J integral, one can write:

$$J = J^{\text{act}} + J^{\text{aux}} + M \quad (4.70)$$

where J corresponds to the superposition state, and J^{act} and J^{aux} are the actual and auxiliary states J integrals, respectively, defined in (2.164)-(2.165), and the interaction integral M is defined in (2.166):

$$M = \int_A \left[\sigma_{ij} \frac{\partial u_i^{\text{aux}}}{\partial x_1} + \sigma_{ij}^{\text{aux}} \frac{\partial u_i}{\partial x_1} - w^M \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA \quad (4.71)$$

where $w^M = 0.5(\sigma_{ij} \varepsilon_{ij}^{\text{aux}} + \sigma_{ij}^{\text{aux}} \varepsilon_{ij})$ for linear elastic conditions (see Eq. (2.169)).

The strain of the auxiliary field can be chosen by either the strain–stress relationship,

$$\varepsilon_{ij}^{\text{aux}} = C_{ijkl} \sigma_{kl}^{\text{aux}} \quad (4.72)$$

or as the symmetric part of the displacement gradient,

$$\varepsilon_{ij}^{\text{aux}} = 0.5(u_{i,j}^{\text{aux}} + u_{j,i}^{\text{aux}}) \quad (4.73)$$

these options are compatible with each other as long as the material is homogeneous.

Auxiliary stress and strain states should be chosen so as to satisfy both the equilibrium equation and the traction free boundary condition on the crack surface in the area A . One of the choices is the displacement and stress fields in the vicinity of the crack tip provided by Sih *et al.* (1965) and Asadpoure *et al.* (2007) in local crack tip coordinates (x, y) (see Section 4.3.1),

$$u_1^{\text{aux}} = K_I^{\text{aux}} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ s_1 p_2 \sqrt{\cos \theta + s_2 \sin \theta} - s_2 p_1 \sqrt{\cos \theta + s_1 \sin \theta} \right\} \right] \\ + K_{II}^{\text{aux}} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ p_2 \sqrt{\cos \theta + s_2 \sin \theta} - p_1 \sqrt{\cos \theta + s_1 \sin \theta} \right\} \right] \quad (4.74)$$

$$u_2^{\text{aux}} = K_I^{\text{aux}} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ s_1 q_2 \sqrt{\cos \theta + s_2 \sin \theta} - s_2 q_1 \sqrt{\cos \theta + s_1 \sin \theta} \right\} \right] \\ + K_{II}^{\text{aux}} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ q_2 \sqrt{\cos \theta + s_2 \sin \theta} - q_1 \sqrt{\cos \theta + s_1 \sin \theta} \right\} \right] \quad (4.75)$$

and

$$\begin{aligned}\sigma_{11}^{aux} = & \frac{K_I^{aux}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{s_1 s_2}{s_1 - s_2} \left\{ \frac{s_2}{\sqrt{\cos \theta + s_2 \sin \theta}} - \frac{s_1}{\sqrt{\cos \theta + s_1 \sin \theta}} \right\} \right] \\ & + \frac{K_{II}^{aux}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ \frac{s_2^2}{\sqrt{\cos \theta + s_2 \sin \theta}} - \frac{s_1^2}{\sqrt{\cos \theta + s_1 \sin \theta}} \right\} \right]\end{aligned}\quad (4.76)$$

$$\begin{aligned}\sigma_{22}^{aux} = & \frac{K_I^{aux}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ \frac{s_1}{\sqrt{\cos \theta + s_2 \sin \theta}} - \frac{s_2}{\sqrt{\cos \theta + s_1 \sin \theta}} \right\} \right] \\ & + \frac{K_{II}^{aux}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ \frac{1}{\sqrt{\cos \theta + s_2 \sin \theta}} - \frac{1}{\sqrt{\cos \theta + s_1 \sin \theta}} \right\} \right]\end{aligned}\quad (4.77)$$

$$\begin{aligned}\sigma_{12}^{aux} = & \frac{K_I^{aux}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{s_1 s_2}{s_1 - s_2} \left\{ \frac{1}{\sqrt{\cos \theta + s_1 \sin \theta}} - \frac{1}{\sqrt{\cos \theta + s_2 \sin \theta}} \right\} \right] \\ & + \frac{K_{II}^{aux}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{1}{s_1 - s_2} \left\{ \frac{s_1}{\sqrt{\cos \theta + s_1 \sin \theta}} - \frac{s_2}{\sqrt{\cos \theta + s_2 \sin \theta}} \right\} \right]\end{aligned}\quad (4.78)$$

After some manipulations (Asadpoure *et al.*, 2007, Asadpoure and Mohammadi, 2007 and Bayesteh and Mohammadi, 2011):

$$J^s = t_{11} (K_I^{aux} + K_I)^2 + t_{12} (K_I^{aux} + K_I) (K_{II}^{aux} + K_{II}) + t_{22} (K_{II}^{aux} + K_{II})^2 \quad (4.79)$$

$$J^{act} = t_{11} (K_I)^2 + t_{12} (K_I K_{II}) + t_{22} (K_{II})^2 \quad (4.80)$$

$$J^{aux} = t_{11} (K_I^{aux})^2 + t_{12} (K_I^{aux} K_{II}^{aux}) + t_{22} (K_{II}^{aux})^2 \quad (4.81)$$

and

$$M = 2t_{11} K_I K_I^{aux} + t_{12} (K_I K_{II}^{aux} + K_I^{aux} K_{II}) + 2t_{22} K_{II} K_{II}^{aux} \quad (4.82)$$

where

$$t_{11} = -\frac{c_{22}}{2} \operatorname{Im} \left(\frac{s_1 + s_2}{s_1 s_2} \right) \quad (4.83)$$

$$t_{12} = -\frac{c_{22}}{2} \operatorname{Im} \left(\frac{1}{s_1 s_2} \right) + \frac{c_{11}}{2} \operatorname{Im} (s_1 s_2) \quad (4.84)$$

$$t_{22} = \frac{c_{11}}{2} \operatorname{Im} (s_1 + s_2) \quad (4.85)$$

The stress intensity factor can then be obtained by considering the two states and solving a system of linear algebraic equations. These two states are state 1: $K_I^{aux} = 1; K_{II}^{aux} = 0$ and state 2: $K_I^{aux} = 0; K_{II}^{aux} = 1$. By calculating M from both Eqs. (4.72) and (4.83) and solving a system of linear algebraic equations, the actual mixed mode stress intensity factors associated with state 1 and state 2 are obtained (Asadpoure *et al.*, 2007 and Asadpoure and Mohammadi, 2007):

$$M^{(1)} = 2t_{11}K_I + t_{12}K_{II} \quad (4.86)$$

$$M^{(2)} = t_{12}K_I + 2t_{22}K_{II} \quad (4.87)$$

4.4.4 Orthotropic crack propagation criteria

In this section, the material behaviour is ideally assumed as a homogeneous orthotropic continuum to simplify the process of crack modelling. As a result, details of local failures of composites, such as cracked matrix, broken fibre, local delamination, etc. are not considered.

4.4.4.1 Maximum circumferential tensile stress criterion

Saouma *et al.* (1987) extended the original isotropic maximum circumferential tensile stress theory to anisotropic solids.

In this case, the fracture toughness is no longer uniquely defined. Instead, two values of K_{Ic}^1 and K_{Ic}^2 along the principal planes of elastic symmetry are required for characterizing the brittle behaviour of the crack in a homogenous transversely isotropic solid with elastic constants E_1, E_2 , and μ_{12} (Fig. 4.4). According to Saouma *et al.* (1987), even for a symmetrically aligned loading and material properties with respect to the crack, a parasitic crack sliding displacement occurs in addition to a pure mode I for anisotropic materials.

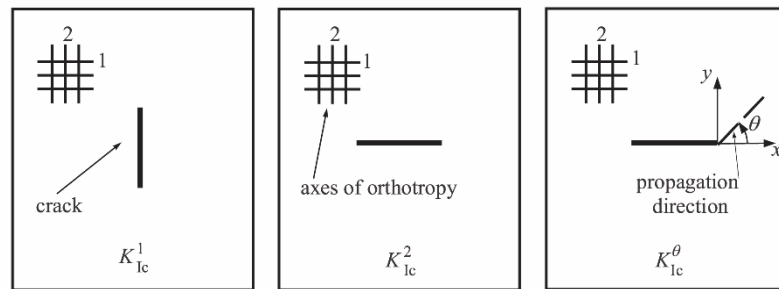


Figure 4.4 Fracture toughness for a homogeneous anisotropic solid (Saouma *et al.*, 1987).

For a crack arbitrarily oriented with respect to direction 1, the toughness K_{Ic}^β can be assumed as a polar function of K_{Ic}^1 and K_{Ic}^2 (Saouma *et al.*, 1987, Ye and Ayari, 1994, Ayari and Ye, 1995):

$$K_{lc}^\theta = K_{lc}^1 \cos^2 \theta + K_{lc}^2 \sin^2 \theta \quad (4.88)$$

where θ is the crack propagation angle with respect to the $x_1 = x$ axis (See Fig. 4.4).

In order to avoid performing two separate fracture toughness tests, it is sometimes assumed that the ratio of the fracture toughness in both directions is equal to the ratio of the elastic modulus (Saouma *et al.*, 1987):

$$K_{lc}^2 = K_{lc}^1 \frac{E_1}{E_2} \quad (4.89)$$

The classical maximum circumferential/hoop tensile stress mode assumes that the crack propagation is along the direction of the maximum tangential stress $\sigma_{\theta\theta}$, while the shear stress is zero. The hoop stress component $\sigma_{\theta\theta}$ for an anisotropic body can be obtained from,

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{s_1 t_1 - s_2 t_2}{s_1 - s_2} \right] + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left[\frac{t_1 - t_2}{s_1 - s_2} \right] \quad (4.90)$$

where

$$t_1^2 = (s_2 \sin \theta + \cos \theta)^3 \quad (4.91)$$

$$t_2^2 = (s_1 \sin \theta + \cos \theta)^3 \quad (4.92)$$

Since the angular variation of material toughness and the critical tensile strength is not constant for anisotropic materials, instead of maximizing the hoop stress component $\sigma_{\theta\theta}$, the ratio of $K_{\theta\theta} / K_{lc}^\theta$ is maximized to determine the angle of crack propagation θ ,

$$\max \left\{ \frac{K_{\theta\theta}}{K_{lc}^\theta} \right\} = \max \left\{ \frac{\sqrt{2\pi r} \sigma_{\theta\theta}}{K_{lc}^\theta} \right\} = \max \left\{ \frac{\left[K_I \operatorname{Re} \left[\frac{s_1 t_1 - s_2 t_2}{s_1 - s_2} \right] + K_{II} \operatorname{Re} \left[\frac{t_1 - t_2}{s_1 - s_2} \right] \right]}{\left[K_{lc}^1 \cos^2 \theta + K_{lc}^2 \sin^2 \theta \right]} \right\} \quad (4.93)$$

4.4.4.2 Minimum strain energy density criterion

It is also possible to extend the minimum strain energy density criterion to anisotropic problems. However, it is important to note that such a criterion is expressed in terms of a scalar quantity (strain energy density), which cannot be directly related to a tensorial (angular dependent) nature of toughness in anisotropic materials (Saouma *et al.*, 1987).

Carloni and Nobile (2002) discussed the extension of classical minimum strain energy density criterion to orthotropic materials. According to this model, propagation occurs at an angle where the ratio of strain energy density to the corresponding material toughness is minimised:

$$\min \left\{ \frac{\frac{dw}{dV}}{\left(\frac{dw}{dV} \right)_c^\theta} \right\} \quad (4.94)$$

where

$$\frac{dw}{dV} = \frac{S}{r} = \frac{1}{r} \left\{ c_{11} K_I^2 + 2c_{12} K_I K_{II} + c_{22} K_{II}^2 \right\} \quad (4.95)$$

$$\left(\frac{dw}{dV} \right)_c^\theta = \left(\frac{dw}{dV} \right)_c^x \sin^2 \theta + \left(\frac{dw}{dV} \right)_c^y \cos^2 \theta \quad (4.96)$$

4.4.4.3 Maximum circumferential strain criterion

Ayari and Ye (1995) have proposed a criterion based on maximisation of the circumferential strain. The method accounts for angular variation of the toughness in orthotropic materials and can reportedly anticipate general mixed mode crack propagations in orthotropic problems. The method leads to rather implicit formulations such as Section 4.4.4.2 than explicit relations of (4.93). For details see Ayari and Ye (1995).

4.4.4.4 Simplified models

Several simplified models are available for specific mixed mode propagation problems. Unfortunately, many of them cannot be used for general orthotropic problems, or at least need further experimental and numerical verifications. Among them, substantial work has been performed on wood and timber materials, which behave in specific orthotropic patterns. For a review of available techniques refer to Jernkvist and Thuvander (2001), and Jernkvist (2001a, 2001b).

4.5 ANISOTROPIC XFEM

The general methodology of isotropic extended finite element can be similarly extended to include anisotropic problems, if anisotropic enrichment functions are embedded into an anisotropic finite element procedure. Generalised Heaviside and near crack tip anisotropic enrichment functions are included as extra degrees of freedom in selected nodes near the discontinuities.

4.5.1 Governing equation

Consider a body in the state of equilibrium with the boundary conditions in the form of traction and displacements that also include a crack (Figure 4.1). The strong form of the equilibrium equation can be written as:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}^b = 0 \quad \text{in } \Omega \quad (4.97)$$

with the following boundary conditions:

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f}^t \quad \text{on } \Gamma_t : \text{external traction} \quad (4.98)$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u : \text{prescribed displacement} \quad (4.99)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_c : \text{traction free crack} \quad (4.100)$$

where Γ_t , Γ_u and Γ_c are traction, displacement and crack boundaries, respectively, $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{f}^b$ and $\mathbf{f}^t$ are the body force and external traction vectors, respectively (Fig. 3.4).

The variational formulation of the boundary value problem can be defined as:

$$W^{\text{int}} = W^{\text{ext}} \quad (4.101)$$

or

$$\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, d\Omega = \int_{\Omega} \mathbf{f}^b \cdot \boldsymbol{\delta} \mathbf{u} \, d\Omega + \int_{\Gamma} \mathbf{f}^t \cdot \boldsymbol{\delta} \mathbf{u} \, d\Gamma \quad (4.102)$$

4.5.2 XFEM discretisation

In the extended finite element method, in order to include the effects of crack surfaces and crack tips, the approximation (3.28) is utilised (ignoring the weak discontinuity term) to calculate the displacement for point $\mathbf{x}$ locating within the domain

$$\begin{aligned} \mathbf{u}^h(\mathbf{x}) = & \left[\sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j \right] + \left[\sum_{h=1}^{mh} N_h(\mathbf{x}) [H(\xi(x)) - H(\xi(x_h))] \mathbf{a}_h \right] \\ & + \left[\sum_{k=1}^{mt} N_k(\mathbf{x}) \left(\sum_{l=1}^{mf} [F_l(x) - F_l(x_k)] \mathbf{b}_k^l \right) \right] \end{aligned} \quad (4.103)$$

where mh is the set of nodes that have crack face (but not crack tip) in their support domain, while mt is the set of nodes associated with the crack tip in their influence domain; $\mathbf{u}_j$ are the nodal displacements (standard degrees of freedom); $\mathbf{a}_h$, $\mathbf{b}_k^l$ are vectors of additional degrees of nodal freedom for modelling crack faces and the crack tip, respectively. $F_l(\mathbf{x})$ represent the crack tip enrichment functions and $H(\mathbf{x})$ is the generalised Heaviside function.

Discretisation of Eq. (4.102) using the XFEM procedure (4.103) results in a discrete system of linear equilibrium equations:

$$\mathbf{K}\mathbf{u}^h = \mathbf{f} \quad (4.104)$$

where $\mathbf{u}^h$ is the vector of degrees of nodal freedom (for both classical and enriched ones),

$$\mathbf{u}^h = \{\mathbf{u} \quad \mathbf{a} \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4\}^T \quad (4.105)$$

The global stiffness matrix $\mathbf{K}$ is assembled from the stiffness of each element Ω^e ,

$$\mathbf{K}_{ij}^e = \begin{bmatrix} \mathbf{K}_{ij}^{uu} & \mathbf{K}_{ij}^{ua} & \mathbf{K}_{ij}^{ub} \\ \mathbf{K}_{ij}^{au} & \mathbf{K}_{ij}^{aa} & \mathbf{K}_{ij}^{ab} \\ \mathbf{K}_{ij}^{bu} & \mathbf{K}_{ij}^{ba} & \mathbf{K}_{ij}^{bb} \end{bmatrix} \quad (4.106)$$

with

$$\mathbf{K}_{ij}^{rs} = \int_{\Omega^e} (\mathbf{B}_i^r)^T \mathbf{D} \mathbf{B}_j^s d\Omega \quad (r, s = \mathbf{u}, \mathbf{a}, \mathbf{b}) \quad (4.107)$$

where $\mathbf{B}_i^u$, $\mathbf{B}_i^a$ and $\mathbf{B}_i^b$ are derivatives of shape functions defined in Eqs. (3.49)-(3.55). $\mathbf{f}$ is the vector of external forces,

$$\mathbf{f}_i^e = \{\mathbf{f}_i^u \quad \mathbf{f}_i^a \quad \mathbf{f}_i^{b1} \quad \mathbf{f}_i^{b2} \quad \mathbf{f}_i^{b3} \quad \mathbf{f}_i^{b4}\}^T \quad (4.108)$$

and

$$\mathbf{f}_i^u = \int_{\Gamma_t} N_i \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i \mathbf{f}^b d\Omega \quad (4.109)$$

$$\mathbf{f}_i^a = \int_{\Gamma_t} N_i [H(\xi) - H(\xi_i)] \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i [H(\xi) - H(\xi_i)] \mathbf{f}^b d\Omega \quad (4.110)$$

$$\mathbf{f}_i^{ba} = \int_{\Gamma_t} N_i (F_\alpha - F_{\alpha i}) \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i (F_\alpha - F_{\alpha i}) \mathbf{f}^b d\Omega \quad (\alpha = 1, 2, 3 \text{ and } 4) \quad (4.111)$$

4.5.3 Orthotropic enrichment functions

Crack tip enrichment functions are obtained from the analytical solution for displacement in the vicinity of the crack tip. The general form of these functions can be defined as,

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta_1}{2} \sqrt{g_1(\theta)}, \sqrt{r} \cos \frac{\theta_2}{2} \sqrt{g_2(\theta)}, \sqrt{r} \sin \frac{\theta_1}{2} \sqrt{g_1(\theta)}, \right. \\ \left. \sqrt{r} \sin \frac{\theta_2}{2} \sqrt{g_2(\theta)} \right\} \quad (4.112)$$

where functions $g_k(\theta)$ and θ_k , in their general form for all orthotropic composites, are defined as:

$$g_k(\theta) = \sqrt{(\cos \theta + s_{kx} \sin \theta)^2 + (s_{ky} \sin \theta)^2} \quad (4.113)$$

$$\theta_k = \arctan \left(\frac{s_{ky} \sin \theta}{\cos \theta + s_{kx} \sin \theta} \right) \quad (4.114)$$

These functions for the type I composites ($a_1 > \sqrt{a_2}$) are

$$g_j(\theta) = \left(\cos^2 \theta + \frac{\sin^2 \theta}{e_j^2} \right)^{1/2} \quad j = 1, 2 \quad (4.115)$$

$$\theta_j = \tan^{-1} \left(\frac{x_2}{e_j x_1} \right) = \tan^{-1} \left(\frac{\tan \theta}{e_j} \right) \quad j = 1, 2 \quad (4.116)$$

where e_1 and e_2 are defined in Eq. (4.51)-(4.52).

Finally, in case of $|a_1| < \sqrt{a_2}$, (type II) the main contributing terms are,

$$g_j(\theta) = \left(\cos^2 \theta + l^2 \sin^2 \theta + (-1)^j l^2 \sin 2\theta \right)^{1/2} \quad j = 1, 2 \quad (4.117)$$

$$\theta_j = \arctan \left(\frac{\gamma_2 l^2 \sin \theta}{\cos \theta + (-1)^j \gamma_1 l^2 \sin \theta} \right) \quad j = 1, 2 \quad (4.118)$$

where γ_1 and γ_2 and l are defined in Eq. (4.55)-(4.57).

According to Eq. (4.112), the first two terms are continuous across the crack faces while the remaining ones are discontinuous.

Eq. (4.112) cannot be directly used in isotropic media because it may lead to indefinite (0/0) expressions (Mohammadi and Asadpoure, 2006). A straightforward remedy is to apply the original isotropic enrichment functions (3.31)

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2} \right\} \quad (4.119)$$